PROJECTED WRITTEN NOTES FROM THE M 408D LECTURE ON TUESDAY, FEBRUARY 6, 2024, on THE LEVEL OF ERROR, SEC. 11.4: THE COMPARISON TEST, and SEC. 11.5: ALTERNATING SERIES.

CLASS #7

Recall that, for now, all the series $\sum_{n=1}^{\infty} a_n$ that we consider have all positive or non-negative terms a_n, i.e., $a_n \geq 0$ for all $n \geq 1$.

Consider these two p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $0 \leq p$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^1}$$

Here, $p = 1 \leq 1$,

So, the series is Divergent as a p-series. It is called the Harmonic Series.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Here, $p = 2 > 1$

So, the series is Convergent as a p-series, since $p > 1$.

It has a summation S and $S = \lim_{n \to \infty} s_n$

Exercise #1:

Using your calculator, for the given sequence $\{a_n\}$ below,

compute the first five terms of its sequence of partial sums $\{s_n\}$.

Fill in the blanks below with the values of s_1, s_2, s_3, s_4 and s_5,

rounded to 6 decimal places.

The Sequence | **Its Sequence of Partial Sums (SOPS)**

$\{a_n\}, n = 1, 2, 3, \ldots$ | $\{s_n\}, n = 1, 2, 3, \ldots$

where $a_n = \dfrac{1}{n^2}$. | where $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k^2}$

n	1/(n^2)	sn
1	1.0000000000	1.0000000000
2	0.2500000000	1.2500000000
3	0.1111111111	1.3611111111
4	0.0625000000	1.4236111111
5	0.0400000000	1.4636111111
6	0.0277777778	1.4913888889
7	0.0204081633	1.5117970522
8	0.0156250000	1.5274220522
9	0.0123456790	1.5397677312
10	0.0100000000	1.5497677312
•	•	•
•	•	•
•	•	•

Each s_n can be used as an approximation to the summation S.

Summation S as $n \to \infty$

s_1 = ____________________ s_2 = ____________________

s_3 = ____________________ s_4 = ____________________

s_5 = ____________________

How good ~~as~~ an approximation of summations is S_n when we "use S_n to approximate S"?

$$\underbrace{\text{use } S_n \text{ to approximate } S}_{S_n \approx S}$$

Answer: FIND the Level of Error.

The Level of ERROR in an approximation:

When A is used to approximate S, the exact error in $A \approx S$ is $|S - A|$

when t is a small number such that we know the ERROR is not greater than t, i.e., ERROR $\leq t$, then t is called "The Level of ERROR in $A \approx S$"

What is really going when we use $S_n \approx S$?

Given the series (Convergent):

$$\sum_{n=1}^{\infty} a_n = S = \underbrace{a_1 + a_2 + a_3 + \cdots + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{\text{The Remainder Series } R_n}$$

$$S = S_n + R_n \Rightarrow R_n = S - S_n = |S - S_n| =$$ The Exact ERROR in $S_n \approx S$.

<u>FACT</u>! When the series $\sum_{k=1}^{\infty} a_k$ is convergent and has sum $S = \sum_{k=1}^{\infty} a_k$, if it has been shown to be convergent by the Integral Test, using the function $f(x)$, then for any positive integer n, where

R_n = the exact error in $S_n \approx S$,

then R_n = Exact ERROR $\leq \int_n^{\infty} f(x)\,dx$ = The Level of ERROR in $S_n \approx S$.

This formula applies to convergent p-series too.

Problem: For the Convergent p-series $\sum_{n=0}^{\infty} \frac{1}{n^5}$,

the integral applies with $f(x) = \frac{1}{x^5}$. (let $n=3$)

What is the Level of Error in using $S_3 \approx S$?

Sol'n: $S_3 = \frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} = 1.035365226$

$R_3 = \frac{1}{4^5} + \frac{1}{5^5} + \frac{1}{6^5} + \cdots$

$S = S_3 + R_3 \Rightarrow R_3 = S - S_3 \approx$ The ERROR in $S_3 \approx S$.

$R_3 \leq \int_3^{\infty} f(x)\,dx = \int_3^{\infty} \frac{1}{x^5}\,dx$ = the Level of ERROR in $S_3 \approx S$.

$\int_3^{\infty} \left(\frac{1}{x^5}\right)dx \overset{\text{Work}}{=} \cdots = \frac{1}{4(3^4)} = 0.00309$

0.00309 is the Level of ERROR in $S_3 \approx S$.

We can show in general that

$$\int_n^{\infty} \frac{1}{x^5}\,dx = \frac{1}{4(n^4)}$$ = the level of Error in $S_n \approx S$.

If the problem asks, "What value of n is such that the ERROR in $S_n \approx S \leq 0.00001$,"

then solve the inequality $\frac{1}{4(n^4)} \leq 0.00001$.

New Idea:

FACTORAL NOTATION $n!$

$0! = 1$, $1! = 1$.

For $n \geq 2$, $n! = n(n-1)(n-2)\times\cdots\times 2\times 1$

So $4! = 4\times 3\times 2\times 1 = 24$

$7! = 7\times(6!) = 7\times 6\times(5!)$

Simplify: $\dfrac{(n+1)!}{(n-2)!} = \dfrac{(n+1)(n)(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$

$= (n+1)(n)(n-1)$.

LEAVE THIS IN FACTORED FORM.

THE COMPARISON TEST AND THE LIMIT COMPARISON TEST

BOTH TESTS ARE TESTS FOR CONVERGENCE OR DIVERGENCE OF SERIES WITH ALL TERMS POSITIVE, $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$ for all n.

BOTH TESTS INVOLVE A SECOND SERIES $\sum_{n=1}^{\infty} b_n$ for which the CONVERGENCE OR DIVERGENCE STATUS IS ALREADY KNOWN! [It may be that the roles of $\sum a_n$ and $\sum b_n$ are reversed!]

THE COMPARISON TEST

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have all terms positive.

Suppose N is some positive integer.

① If $a_n \leq b_n$ for all $n \geq N$,
and if $\sum_{n=1}^{\infty} b_n$ is convergent,
then $\sum_{n=1}^{\infty} a_n$ is convergent, too.

So, a series dominated by a convergent series is a convergent series.

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② If $a_n \leq b_n$ for all $n \geq N$,
and if $\sum_{n=1}^{\infty} a_n$ is divergent,
then $\sum_{n=1}^{\infty} b_n$ is divergent, too.

So, a series which dominates a divergent series is a divergent series.

The (Direct) Comparison TEST

EXAMPLE: Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2+6} = \sum_{n=1}^{\infty} c_n$ C or D?

Sol'n: $c_n = \frac{1}{n^2+6}$ compares with $\frac{1}{n^2}$.

So we use $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is C.

First REQ'D Sentence: "The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series which $p = 2$ and $2 > 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is Convergent."

Since $n^2+6 \geq n^2$, $\frac{1}{n^2+6} \leq \frac{1}{n^2}$ for all $n \geq 1$.

Second Req'd Sentence: "Because (1) $\frac{1}{n^2+6} \leq \frac{1}{n^2}$ for all $n \geq 1$ and (2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is Convergent, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+6}$ is Convergent by the (Direct) Comparison Test."

NEW IDEA:

We will now allow some of the a_n terms in $\sum_{n=1}^{\infty} a_n$ to be negative.

ALTERNATING SERIES $\sum_{n=1}^{\infty} a_n$, with

the convention that $b_n = |a_n|$,

are series with one of two forms:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

OR

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

An example:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ when } b_n = \left|(-1)^{n+1}\frac{1}{n}\right| = \frac{1}{n}$$

for $n \geq 1$.

The ALTERNATING SERIES TEST FOR CONVERGENCE

If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$

where $b_n = |a_n|$

satisfies

① $b_{n+1} \le b_n$ for all $n \geq 1$

and

② $\lim_{n\to\infty} b_n = 0$,

then the Alternating Series $\sum_{n=1}^{\infty} a_n$ is Convergent.

(Note: If $\lim_{n\to\infty} b_n \neq 0$ or $\lim_{n\to\infty} b_n$ DNE,

then you can use the Test for Divergence to show that $\sum_{n=1}^{\infty} a_n$ is D.)

Ex: Consider the Alternating Harmonic Series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

where $b_n = \frac{1}{n}$.

We apply the ALTERNATING SERIES TEST:

Step (i): Show that $b_{n+1} \le b_n$ for all $n \geq 1$.

Since $n+1 \geq n$, $b_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = b_n$.

So, $b_{n+1} \le b_n$ for all $n \geq 1$.

Step (ii) Show that $\lim_{n\to\infty} b_n = 0$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

"Because ① $\underbrace{b_{n+1} \le b_n}$ for all $n \geq 1$ ($\frac{1}{n+1} \le \frac{1}{n}$)

and ② $\lim_{n\to\infty} \frac{1}{n} = 0$,

The Alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is

Convergent by the Alternating Series Test"